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THE SLOW ROTATION OF AN AXISYMMETRIC SOLID SUBMERGED IN A FLUID WITH A SURFACTANT SURFACE LAYER—I

THE ROTATING DISK IN A SEMI-INFINITE FLUID

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Abstract—A formulation in terms of a Fredholm integral equation of the first kind is given for the axisymmetric problem of a solid rotating in a bounded viscous fluid whose surface is contaminated with an immiscible surfactant film. The particular case ofa rotating thin circular disk immersed in a semi-infinite body of fluid is studied in detail, the problem being reduced to the solution of a Fredholm integral equation of the second kind. This equation is solved both asymptotically and numerically, and the resistive torque on the disk and surface velocity profiles are computed for varying values of the ratio of the coefficient of surface shear viscosity to the coefficient of viscosity of the substrate fluid, and depth of the disk below the surface.

1. INTRODUCTION

For many years chemists, chemical engineers and others have been interested in the two-phase fluid system in which a surfactant monomolecular film or fluid covers the surface of a large body of different fluid. For incompressible insoluble surfactants the effect on the dynamics of the substrate fluid is described by means of a coefficient of surface shear viscosity η, which appears in the boundary conditions applied at the substrate-surfactant surface. The appropriate boundary condition can be deduced from the work of Scriven (1960), who studied the motion of a thin fluid interface between two bulk fluids of differing viscosities. Thus, if the substrate fluid is in axisymmetrical rotational motion in which the fluid particles have only an azimuthal component of velocity $v(\rho, z)$, (ρ, z) being cylindrical polar coordinates with the z-axis drawn perpendicular to the film and into the bulk fluid, the conventional condition $\partial v/\partial z = 0$ at a free surface is replaced by the condition

$$\mu \frac{\partial v}{\partial z} - \eta \frac{\partial^2 v}{\partial z^2} = 0$$

at the surfactant surface, where μ is the substrate coefficient of viscosity.

A number of ways have been suggested for the experimental determination of η, and a review of recent work is given in Goodrich (1973). In particular, this author and his collaborators (1969, 1970), and more recently Briley *et al.* (1976) and the present author (Shail 1978) have examined the dynamics of a rotating disk viscometer for the measurement of η. The apparatus consists of a thin disk inserted into the plane interface between the surfactant film and the underlying substrate. The disk is rotated slowly, and the torque necessary to maintain the steady rotation is measured. Assuming that the fluid motion is slow enough to permit linearisation of the Navier–Stokes equations, suitable theoretical formulae for the driving torque in terms of the ratio η/μ enable the surface shear viscosity to be evaluated.

This apparatus appears to have several drawbacks. The theoretical analyses indicate that it is not a particularly sensitive method for measuring small coefficients of shear viscosity. Further, from the practical point of view the delicate positioning of the disk in the surfactant-substrate interface would seem to be of some difficulty. Thus, it is of interest to examine alternative configurations and the object of this series of papers is to consider some viscometry problems for rotating bodies which are *completely immersed* in the substrate fluid, a system which also avoids the evaluation of film torques such as act on the edge of the disk in the Goodrich apparatus. This type of instrument can be regarded as the axisymmetric analogue of the viscous traction surface viscometer (see Goodrich 1973).

The outline of this paper is as follows. In section 2 the axisymmetric problem of a slowly rotating solid in a container of viscous fluid is formulated. The plane fluid surface is covered with a surfactant film. A Green's function type integral representation is derived for the azimuthal velocity component $v(\rho, z)$, which satisfies the boundary conditions on the wetted surface of the container and the surfactant film (the ratio $\lambda = \eta/\mu$ enters into this representation via the Green's function). Imposition of the no-slip boundary condition on the rotating solid then leads to a Fredholm integral equation of the first kind for the source distribution in the integral representation for $v(\rho, z)$. The resistive torque is expressed in terms of this source density.

In section 3 the problem is specialised to that of an immersed thin circular disk rotating in a semi-infinite fluid. Using Williams' (1962) method the integral equation of the first kind, whose solution has an integrable singularity at the disk edge, is converted into a Fredholm integral equation of the second kind for a derived quantity $\theta(x)$ which is regular on $[0, 1]$, its domain of definition. The torque is also expressed in terms of $\theta(x)$. This integral equation does not seem to be soluble in closed form, so in section 4 some asymptotic results are derived, valid when h, the depth of the plane of the disk beneath the surface, is large compared with the radius of the disk.

Section 5 contains the results of a numerical investigation of the integral equation for $\theta(x)$ and the evaluation of the resistive torque on the disk for varying values of λ and h.

It has been suggested by Prof. Howard Brenner that, rather than measure the driving torque on the disk, a better experimental procedure for determining η would be to measure the azimuthal velocity, or equivalently the period of revolution, of a suitably marked fluid particle in the surfactant fluid surface. Thus, in section 6 the problem of computing surface velocity profiles is investigated, and some numerical and graphical results are presented.

2. BASIC FORMULATION

The geometrical configuration considered is as follows. An axisymmetric container, whose axis is vertical, contains incompressible viscous fluid on whose plane horizontal surface is a thin layer of immiscible surfactant a few molecules thick. We denote by S_1 the wetted surface of the container and by S_2 the surfactant layer. The fluid contains a *fully-immersed* axisymmetric solid which rotates slowly about the common axis of symmetry of the solid and the container with constant angular speed Ω. The surface of the rotating solid is S, and T is the bulk fluid volume bounded by S and $S_1 \cup S_2$.

Let O be a suitably chosen origin and let (ρ, ϕ, z) be cylindrical polar coordinates such that Oz coincides with the downward drawn vertical axis of symmetry of the configuration. The surfactant layer then occupies a circular region S_2 of radius R of the plane $z = -h$, where $h > 0$. For sufficiently small angular Reynolds numbers the velocity field $\mathbf{v}$ in the bulk fluid satisfies the linearized Navier–Stokes and continuity equations

$$\mu \operatorname{curl} \operatorname{curl} \mathbf{v} = -\nabla p, \tag{1}$$

$$\operatorname{div} \mathbf{v} = 0, \tag{2}$$

where p is the dynamic pressure and μ the coefficient of viscosity of the bulk fluid. Assuming that the streamlines are circles lying in planes perpendicular to Oz, the vector $\mathbf{v}$ has a non-zero component $v(\rho, z)$ in the ϕ-direction only. Since v is independent of ϕ, the equation of continuity (2) is satisfied identically, and from (1) it follows that p is constant throughout the fluid, v satisfying the equation

$$\frac{\partial^2 v}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial v}{\partial \rho}+\frac{\partial^2 v}{\partial z^2}-\frac{v}{\rho^2}=0. \qquad [3]$$

The boundary conditions imposed on v are the usual no-slip conditions on S and S_1, and on the surfactant region S_2 the balance of substrate stresses and the internal film stresses leads to the "generalized impedance condition" discussed in the introduction to this paper. Thus,

$$v(\rho, z)=\Omega\rho, \quad (\rho, z)\in S, \qquad [4]$$

$$v(\rho, z)=0, \quad (\rho, z)\in S_1, \qquad [5]$$

and

$$\mu\frac{\partial v}{\partial z}-\eta\frac{\partial^2 v}{\partial z^2}=0, \quad (\rho, z)\in S_2 \qquad [6]$$

($z=-h$ on S_2), where η is the coefficient of surface shear viscosity of the adsorbed film. In terms of $\lambda=\eta/\mu$, [6] reads

$$\frac{\partial v}{\partial z}-\lambda\frac{\partial^2 v}{\partial z^2}=0, \quad (\rho, z)\in S_2. \qquad [6a]$$

Thus $\lambda=0$ corresponds to a clean surface. When $\lambda\to\infty$, [6a] is equivalent to $\partial^2 v/\partial z^2=0$ on $z=-h$. Using [3], $v(\rho,-h)$ satisfies

$$\frac{\partial^2 v}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial v}{\partial \rho}-\frac{v}{\rho^2}=0,$$

with general solution

$$v(\rho,-h)=A\rho+B\rho^{-1}.$$

Since $v(0,-h)=v(R,-h)=0$ it follows that $A=B=0$. Thus, in this limit, $v(\rho,-h)=0$ and the surfactant acts as a rigid plane boundary.

We next construct an integral representation for $v(\rho, z)$ by utilising Green's theorem. From [3] it is easily verified that the function

$$w(\rho,\phi,z)=v(\rho,z)\cos\phi$$

is harmonic in T, and clearly satisfies conditions [5] and [6] if v does. Let $G(\mathbf{r},\mathbf{r}')$ be a Green's function which satisfies [5] and [6] and the equation

$$\nabla^2 G(\mathbf{r},\mathbf{r}')=-4\pi\delta(\mathbf{r}-\mathbf{r}')$$

for $\mathbf{r}$, $\mathbf{r}'\in T$. An application of Green's theorem to the function w and G in the region T bounded by $S\cup S_1\cup S_2$ now shows that, for $\mathbf{r}\in T$,

$$w(\mathbf{r})=-\frac{1}{4\pi}\int_S\int G(\mathbf{r},\mathbf{r}')\frac{\partial w}{\partial n'}(\mathbf{r}')\,dS'+\frac{1}{4\pi}\int_S\int \rho'\cos\phi'\frac{\partial G}{\partial n'}(\mathbf{r},\mathbf{r}')\,dS'$$
$$+\frac{1}{4\pi}\int_{S_2}\int\left\{G(\mathbf{r},\mathbf{r}')\frac{\partial w}{\partial z'}(\mathbf{r}')-w(\mathbf{r}')\frac{\partial G}{\partial z'}(\mathbf{r},\mathbf{r}')\right\}dS', \qquad [7]$$

where $\partial/\partial n'$ denotes differentiation along the outward drawn normal to S.

On S_2, $\mathrm{d}S' = \rho'\,\mathrm{d}\rho'\,\mathrm{d}\phi'$ and, omitting arguments of functions for brevity, the ρ'-integration in the final integral in [7] can be written as

$$\lambda \int_0^R \left\{ G^{(1)} \frac{\partial^2 v}{\partial z'^2} - v\,\frac{\partial^2 G^{(1)}}{\partial z'^2} \right\} \rho'\,\mathrm{d}\rho', \qquad [8]$$

where $G^{(1)}(\rho, z; \rho', z')$ is the coefficient of $\cos(\phi - \phi')$ in the Fourier expansion of $G(\mathbf{r}, \mathbf{r}')$. $G^{(1)}(\rho, z; \rho', z')$ satisfies [3] for $\rho \neq \rho'$, $z \neq z'$, and hence [8] can be expressed as

$$\lambda \int_0^R \left\{ G^{(1)} \left(\frac{\partial v}{\partial \rho'} + \rho' \frac{\partial^2 v}{\partial \rho'^2} \right) - v \left(\frac{\partial G^{(1)}}{\partial \rho'} + \rho' \frac{\partial^2 G^{(1)}}{\partial \rho'^2} \right) \right\} \mathrm{d}\rho'. \qquad [9]$$

The vanishing of integral [9] now follows by several integrations by parts and use of the conditions $v(R, -h) = G^{(1)}(\rho,\ z;\ R,\ -h) = 0$ and $v(0, -h) = 0$. Note that if the rotating solid intersects the surface S_2, [9] is not identically zero, there being non-vanishing contributions to the parts integrations from the lower limit, which is no longer zero. (See Shail (1978) for an alternative formulation of such a problem for a rotating disk in the surface.)

In order to simplify [7] still further we next apply Green's theorem to the functions $\rho' \cos \phi'$ and $G(\mathbf{r}, \mathrm{r}')$ in the region T^* interior to S with $\mathbf{r} \in T$, to obtain

$$\int_S\!\!\int \rho' \cos \phi' \frac{\partial G}{\partial n'}\,\mathrm{d}S' = \int_S\!\!\int G(\mathbf{r}, \mathbf{r}') \frac{\partial}{\partial n'} (\rho' \cos \phi')\,\mathrm{d}S'. \qquad [10]$$

Thus, combining [7], the vanishing of [9], and [10], it follows that

$$w(\mathbf{r}) = \int_S\!\!\int \sigma(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') \cos \phi'\,\mathrm{d}S', \quad \mathbf{r}' \in S, \quad \mathbf{r} \in T, \qquad [11]$$

where the source density $\sigma(\mathbf{r}')$ is defined by

$$\sigma(\mathbf{r}') = -\frac{1}{4\pi} \rho' \frac{\partial}{\partial n'} \left(\frac{v}{\rho'} \right). \qquad [12]$$

Condition [4] now gives a Fredholm integral equation of the first kind for the determination of σ, namely

$$\Omega \rho \cos \phi = \int_S\!\!\int \sigma(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') \cos \phi'\,\mathrm{d}S', \quad \mathbf{r}, \mathbf{r}' \in S',$$

or, equivalently,

$$\Omega \rho = \pi \int_C \sigma(\rho', z') G^{(1)}(\rho, z; \rho', z') \rho'\,\mathrm{d}l', \quad (\rho', z') \in C, \qquad [13]$$

where C is the bounding curve of S in a meridional plane, and $\mathrm{d}l'$ denotes the element of arc length of C.

If the rotating solid is a thin circular disk of radius a in the plane $z = 0$, then [10] is meaningless. However, it follows directly from [7] that [13] is still valid, the integration being along the upper and lower edges of the meridian section of the disk; defining $\sigma^* = 2\sigma$, then [13] may be written as

$$\Omega \rho = \pi \int_0^a \sigma^*(\rho', 0) G^{(1)}(\rho, 0; \rho', 0) \rho'\,\mathrm{d}\rho', \quad 0 \leq \rho \leq a. \qquad [14]$$

Consider next the frictional couple N acting on the rotating solid. The tangential shear stress τ on the surface S in the direction of ϕ-increasing is

$$\tau = \mu\rho \frac{\partial}{\partial n}\left(\frac{v}{\rho}\right)$$
$$= -4\pi\mu\sigma(\rho, z).$$

Thus,

$$N = \mu \int_S\int \rho^2 \frac{\partial}{\partial n}\left(\frac{v}{\rho}\right) \mathrm{d}S$$
$$= -8\pi^2\mu \int_C \rho^2\sigma(\rho, z)\,\mathrm{d}l. \qquad [15]$$

In the case of the rotating disk, the appropriate couple formula is

$$N = -8\pi^2\mu \int_0^a \rho^2\sigma^*(\rho, 0)\,\mathrm{d}\rho. \qquad [16]$$

3. THE ROTATING DISK PROBLEM

In the remaining sections of this paper we treat in detail the problem of a slowly rotating thin disk in a *semi-infinite* fluid occupying the region $-h \le z < \infty$. Units are chosen so that the radius of the disk is unity, it being specified by $z = 0$, $0 \le \rho \le 1$, $0 \le \phi \le 2\pi$. The appropriate Green's function now satisfies [6] on $z = -h$, $0 \le \rho < \infty$, $0 \le \phi \le 2\pi$ and, together with its first derivatives, must vanish as $\rho^2 + z^2 \to \infty$. The method of separation of variables shows that

$$G^{(1)}(\rho, z; \rho', z') = 2\int_0^\infty J_1(t\rho)J_1(t\rho')\left\{\mathrm{e}^{-t|z-z'|} + \frac{1-\lambda t}{1+\lambda t}\,\mathrm{e}^{-2th-t(z+z')}\right\}\mathrm{d}t. \qquad [17]$$

In [17] the term

$$2\int_0^\infty J_1(t\rho)J_1(t\rho')\,\mathrm{e}^{-t|z-z'|}\,\mathrm{d}t$$

is recognisable as the coefficient of $\cos(\phi - \phi')$ in the expansion of the singular inverse distance Green's function in cylindrical coordinates and the remaining term in [17], which is regular when $\rho = \rho'$, $z = z'$, is a correction to ensure that [6] is satisfied on $z = -h$.

The governing Fredholm integral equation [14] can be written as

$$\int_0^1 f(\rho')K_0(\rho, \rho')\,\mathrm{d}\rho' = \rho - \int_0^1 f(\rho')\left\{\int_0^\infty \frac{1-\lambda t}{1+\lambda t}\,\mathrm{e}^{-2ht}\,J_1(t\rho)J_1(t\rho')\,\mathrm{d}t\right\}\mathrm{d}\rho', \quad 0 \le \rho \le 1, \qquad [18]$$

where

$$K_0(\rho, \rho') = \int_0^\infty J_1(t\rho)J_1(t\rho')\,\mathrm{d}t \qquad [19]$$

and

$$\Omega f(\rho) = 2\pi\rho\sigma^*(\rho, 0). \qquad (20)$$

It is well-known that as $h \to \infty$ the solution $f(\rho')$ of [18] exhibits an inverse square root singularity as $\rho' \to 1 - 0$. In order to carry out numerical analysis for arbitrary λ, h, it is therefore convenient to transform [18] into a Fredholm integral equation of the second kind for a derived function which is regular. This is a routine calculation using Williams' (1962) method, and the result is

$$\theta(x)+\int_0^1 L(x, y)\theta(y)\,\mathrm{d}y = 2x, \quad 0 \le x \le 1, \tag{21}$$

where

$$\theta(x) = x\int_x^1 \frac{w^{-1}f(w)}{(w^2-x^2)^{1/2}}\,\mathrm{d}w \tag{22}$$

and

$$L(x, y) = \frac{2}{\pi}\int_0^\infty \frac{1-\lambda t}{1+\lambda t}\,\mathrm{e}^{-2ht}\sin tx \sin ty\,\mathrm{d}t. \tag{23}$$

In order to express the frictional couple in terms of $\theta(x)$ we note that from [22],

$$f(w) = -\frac{2w}{\pi}\frac{\mathrm{d}}{\mathrm{d}w}\int_w^1 \frac{\theta(x)}{(x^2-w^2)^{1/2}}\,\mathrm{d}x. \tag{24}$$

Substituting in [16], using [20], then gives

$$N = 8\Omega\mu\int_0^1 \rho^2\frac{\mathrm{d}}{\mathrm{d}\rho}\left\{\int_\rho^1 \frac{\theta(x)}{(x^2-\rho^2)^{1/2}}\,\mathrm{d}x\right\}\mathrm{d}\rho;$$

on integrating by parts and interchanging orders of integration the result

$$N = -16\Omega\int_0^1 x\theta(x)\,\mathrm{d}x \tag{25}$$

is obtained. We note here that in the limit $h\to\infty$, i.e. an unbounded fluid, [21] has solution

$$\theta(x) = 2x, \quad 0\le x\le 1,$$

and from [25],

$$N_\infty = -\frac{32}{3}\Omega\mu,$$

a well-known result. Further, from [24],

$$f(w) = \frac{4}{\pi}\frac{w^2}{(1-w^2)^{1/2}},$$

exhibiting the inverse square root singularity referred to above.

For arbitrary h and λ the solution of integral equation [21] is not known and hence numerical or asymptotic procedures must be used. However the kernel [23] can be evaluated in terms of tabulated functions, a feature which is useful in checking computed values in the numerical solution of [21].

The kernel [23] can be written in the form

$$L(x, y) = \frac{1}{\pi}\{M(|x-y|) - M(x+y)\}, \tag{26}$$

where

$$M(v) = \int_0^\infty \frac{1-\lambda t}{1+\lambda t}\,\mathrm{e}^{-2ht}\cos vt\,\mathrm{d}t = -\int_0^\infty \mathrm{e}^{-2ht}\cos vt\,\mathrm{d}t + 2\int_0^\infty \frac{1}{1+\lambda t}\,\mathrm{e}^{-2ht}\cos vt\,\mathrm{d}t. \tag{27}$$

The first integral in [27] is elementary, and the second can be expressed in terms of the exponential integral $E_1(z)$ (see Abramowitz & Stegun 1964), defined for complex z with $\operatorname{Re} z > 0$ by

$$E_1(z) = \int_1^\infty t^{-1} e^{-zt} \, dt.$$

The final form obtained for [27] is

$$M(v) = -\frac{2h}{4h^2 + v^2} + \frac{2e^{2h/\lambda}}{\lambda} \operatorname{Re} e^{iv/\lambda} E_1\{(2h + iv)/\lambda\}. \qquad [28]$$

A short table of $E_1(z)$ for complex values of the argument z is given in Abramowitz & Stegun (1964).

4. ASYMPTOTIC RESULTS

When $h \gg 1$ the interaction of the disk with the surfactant is weak and it is possible to obtain iterative solutions to [21]. We delineate three distinct cases wherein (i) $\lambda = O(h)$, i.e. the surfactant film is very viscous compared with the bulk fluid, (ii) $\lambda = O(h^{-1})$, i.e. the film has small surface shear viscosity compared with the viscosity of the bulk fluid, and (iii) $\lambda = O(1)$.

4.1 *The case* $h \gg 1, \lambda = O(h)$

We write $q = \lambda/h$, whence $q = O(1)$, and put $ht = u$ in [23]. The asymptotic expansion of $L(x, y)$ is then found as

$$L(x, y) = \frac{2}{\pi h} \int_0^\infty \frac{1 - qu}{1 + qu} e^{-2u} \left\{ \frac{xyu^2}{h^2} - \frac{u^4}{6h^4}(x^3y + xy^3) + O(h^{-6}) \right\} du,$$

and the first terms in the Neumann iterative solution of [21] are

$$\theta(x) = 2x\left(1 - \frac{2}{3\pi h^3} \int_0^\infty \frac{1 - qu}{1 + qu} u^2 e^{-2u} \, du\right) + O(h^{-5}). \qquad [29]$$

The infinite integral in [29] can be evaluated in terms of exponential integrals with the result that

$$\int_0^\infty \frac{1 - qu}{1 + qu} u^2 e^{-2u} \, du = \frac{2}{q^3} e^{2/q} E_1(2/q) - \frac{1}{q^2} + \frac{1}{2q} - \frac{1}{4}. \qquad [30]$$

Using [29] in [25] an approximation to the torque ratio N/N_∞ is

$$\frac{N}{N_\infty} = 1 - \frac{2}{3\pi h^3} \left\{ \frac{2}{q^3} e^{2/q} \, E_1(2/q) - \frac{1}{q^2} + \frac{1}{2q} - \frac{1}{4} \right\} + O(h^{-5}). \qquad [31]$$

4.2 *The case* $h \gg 1, \lambda = O(h^{-1})$

Setting $\lambda h = r$, where $r = O(1)$, the function $M(v)$ in [28] can be written as

$$M(v) = -\frac{2h}{4h^2 + v^2} + \frac{2h}{r} \operatorname{Re} e^{(2h^2/r) + (ivh/r)} E_1\left(\frac{2h^2}{r} + \frac{ivh}{r}\right).$$

For $h \gg 1$, $M(v)$ is now expanded asymptotically using the result that

$$e^z E_1(z) \sim \frac{1}{z} - \frac{1}{z^2} + \frac{2}{z^3} - \frac{6}{z^4} + \cdots,$$

as $|z| \to \infty$ for $|\arg z| < (3/2)\pi$, and the form obtained for the kernel [26] is

$$L(x, y) = \frac{xy}{2\pi h^3}\left\{1 - \frac{1}{2h^2}(x^2 + y^2 + 6r)\right\} + O(h^{-7}). \qquad [32]$$

The iterative solution of [21] and the torque ratio N/N_∞ are found as

$$\theta(x) = 2x - \frac{x}{3\pi h^3} + \frac{1}{2\pi h^5}\left(\frac{1}{3}x^3 + \frac{1}{5}x + 2rx\right) + O(h^{-6}), \qquad [33]$$

and

$$\frac{N}{N_\infty} = 1 - \frac{1}{6\pi h^3} + \frac{1}{2\pi h^5}\left(\frac{1}{5} + r\right) + O(h^{-6}). \qquad [34]$$

As might be expected, the effect of the weak surfactant only appears in [34] in the term of order h^{-5}, whereas for the very viscous surfactant its effect is apparent in a lower-order term in [29].

4.3 *The case* $h \gg 1, \lambda = O(1)$

In this case the integral in form [23] of the kernel can be expanded asymptotically in inverse powers of h using Watson's lemma, with the result that

$$L(x, y) = \frac{1}{2\pi h^3}\left\{xy - \frac{3\lambda xy}{h}\right\} + O(h^{-5}). \qquad [35]$$

The Neumann iterative solution of [21] is

$$\theta(x) = 2x - \frac{x}{3\pi h^3} + \frac{\lambda x}{\pi h^4} + O(h^{-5}), \qquad [36]$$

and the torque ratio is

$$\frac{N}{N_\infty} = 1 - \frac{1}{6\pi h^3} + \frac{\lambda}{2\pi h^4} + O(h^{-5}). \qquad [37]$$

The effect of the surfactant appears in [37] in the term of order h^{-4}.

All the approximate formulae of this section may be generalised without difficulty to higher powers of h^{-1}. In some cases their range of applicability can be extended using the various methods of series improvement suggested by Van Dyke (1974).

5. NUMERICAL SOLUTION OF [21]

In this section we outline a numerical treatment of integral equation [21] using the standard Fox–Goodwin (1953) procedure. This method replaces [21] by a set of n simultaneous linear equations for the unknown $\theta(x)$ evaluated at a selected number n of equally spaced pivotal points on the range [0, 1]. The finite integral in [21] is approximated by means of Gregory's integration formula, which involves a difference correction containing both forward and backward differences to the trapezoidal rule (this integration formula has the advantage of requiring function evaluations only at pivotal points inside the range of integration). Up to sixth differences were included in our computations. A first approximation to the solution of the approximating set of linear equations with differences is found by neglecting the difference correction. This first approximation is then combined with the difference correction in an iterative procedure to calculate more accurate solutions to the linear equations. The iterative cycle is repeated until, in a suitable norm, successive iterates differ by less than a preassigned amount ϵ. (In our calculations ϵ was taken as 10^{-5}.) Having calculated θ at the n pivotal points, the torque on the rotating disk is computed from [25] by Simpson's rule.

The procedure adopted was to start with $n = 7$ and calculate the solution $\theta(x)$ and torque for the given h and λ. n was then increased in steps of 4 or 6 until successive values of the torque showed no change correct to three decimal places. It was found that apart from cases where h was small (e.g. $h = 0.05$), the iterative process converged in no more than two or three cycles for each value of n, and it was rarely necessary to go beyond $n = 15$.

In the formulation of the set of linear equations described above, it is necessary to compute the kernel [23] at pivotal values of x and y. This was carried out using a NAG library integration routine of the Patterson type, but in order to obtain satisfactory convergence it proved convenient to use two forms of the integral M appearing in [26] and [27]. A simple contour integral transformation shows that

$$M(v) = \int_0^\infty \frac{1-\lambda t}{1+\lambda t} \mathrm{e}^{-2ht} \cos vt \, \mathrm{d}t = \int_0^\infty \frac{1-\lambda^2 t^2}{1+\lambda^2 t^2} \mathrm{e}^{-vt} \sin 2ht \, \mathrm{d}t + 2\lambda \int_0^\infty \frac{t}{1+\lambda^2 t^2} \mathrm{e}^{-vt} \cos 2ht \, \mathrm{d}t, \qquad [38]$$

where the various parameters satisfy the inequalities $0 \le v \le 2$, $0 < h < \infty$, $0 \le \lambda < \infty$. Writing $s = v/2h$, the form

$$M(v) = \frac{1}{2h} \int_0^\infty \frac{4h^2 - \lambda^2 t^2}{4h^2 + \lambda^2 t^2} \mathrm{e}^{-st} \sin t \, \mathrm{d}t + 2\lambda \int_0^\infty \frac{t}{4h^2 + \lambda^2 t^2} \mathrm{e}^{-st} \cos t \, \mathrm{d}t \qquad [39]$$

was used for $s \ge 1$, whereas for $0 < s < 1$ the original expression

$$M(v) = \frac{1}{v} \int_0^\infty \frac{v - \lambda t}{v + \lambda t} \mathrm{e}^{-t/s} \cos t \, \mathrm{d}t \qquad [40]$$

was preferred. In [39] the infinite range of integration was truncated at $t = 18/s$, the corresponding limit in [40] being $18s$. $M(0)$ was approximated by

$$M(0) = \frac{1}{2h} \int_0^{18} \frac{2h - \lambda t}{2h + \lambda t} \mathrm{e}^{-t} \, \mathrm{d}t. \qquad [41]$$

Sample computed values of $L(x, y)$ for various x, y, λ, h were checked against values calculated using [26], [28] and the tables in Abramowitz & Stegun (1964); in each case the agreement was complete to five decimal places.

Detailed calculations were made for varying λ and $h = 0.25$, i.e. the disk at a depth below the surfactant film of one quarter of its radius. Table 1 gives the computed values of $N(\lambda)/\mu\Omega$, where $N(\lambda)$ is the resistive torque, and also the ratio $N(\lambda)/N(0)$ of the torque with the surfactant present to the torque when $z = -h$ is a clean free surface. The values given are correct to 3 decimal places.

We observe from table 1 that the increase in $N(\lambda)/N(0)$ is most rapid for surfactants with small λ; there is only a 1.2 per cent increase in the ratio as λ goes from 100 to ∞. This indicates that the submerged-disk apparatus is more suitable for measuring small values of λ, whereas the Goodrich configuration, involving direct edge interaction with the film, is more sensitive for larger λ.

To evaluate λ for an observed value of $N(\lambda)$, several procedures can be used, e.g. inverse interpolation. Thus, from the values of $N(\lambda)$ for $\lambda = 0.1(0.1)0.6$ with $h = 0.25$, the value of λ corresponding to $N(\lambda) = 9.7630$ is given by inverse interpolation as 0.34995; the actual value is 0.350! Alternatively, the method of least squares can be used to find a polynomial representation of λ as a function of $N(\lambda)$. For $h = 0.25$ and $0 \le \lambda \le 1$ such a fit, accurate to about 1 per

Table 1.

λ	0	0.25	0.5	0.75	1	1.25
$N(\lambda)/\mu\Omega$	8.241	9.565	10.421	11.038	11.509	11.882
$N(\lambda)/N(0)$	1	1.161	1.265	1.340	1.397	1.442
λ	1.5	1.75	2	3	4	5
$N(\lambda)/\mu\Omega$	12.186	12.439	12.653	13.259	13.638	13.898
$N(\lambda)/N(0)$	1.479	1.510	1.536	1.609	1.655	1.687
λ	6	8	10	50	100	∞
$N(\lambda)/\mu\Omega$	14.088	14.347	14.516	15.142	15.231	15.323
$N(\lambda)/N(0)$	1.710	1.741	1.762	1.838	1.848	1.860

cent, is given by

$$\lambda = -7.51406 + 2.462503N - 0.285236N^2 + 0.011776N^3.$$

Note, however, that the regression coefficients in such a formula will vary with h, and to avoid this multiple regression must be used.

To see the effect of decreasing h, table 2 gives some values for $h = 0.125$. Comparing with table 1, it will be seen that there is, for example, a 63 per cent increase in the ratio $N(\lambda)/N(0)$ for $h = 0.125$, $\lambda = 0.75$ against the 34 per cent increase when $h = 0.25$. In order to emphasize the torque, the disk should be placed as near to the surface as is practicable; however it should not be so close to the surface that finite disk-thickness effects begin to distort the results.

The variation of $N(\lambda)/N(0)$ with λ is shown in figure 1 for $h = 0.25$ and $0 \le \lambda \le 1$; for larger λ the curve approaches asymptotically the value 1.860 corresponding to $\lambda \to \infty$ (a rigid plane).

It was found that for very small h convergence of the Fox–Goodwin scheme was slow, particularly for large λ (this is probably düe to the same ill-conditioning of the approximating set of linear equations as observed by Fox and Goodwin in their investigation of Love's

Table 2.

λ	0	0.25	0.5	0.75
$N(\lambda)/\mu\Omega$	7.177	9.167	10.570	11.662
$N(\lambda)/N(0)$	1	1.277	1.473	1.625

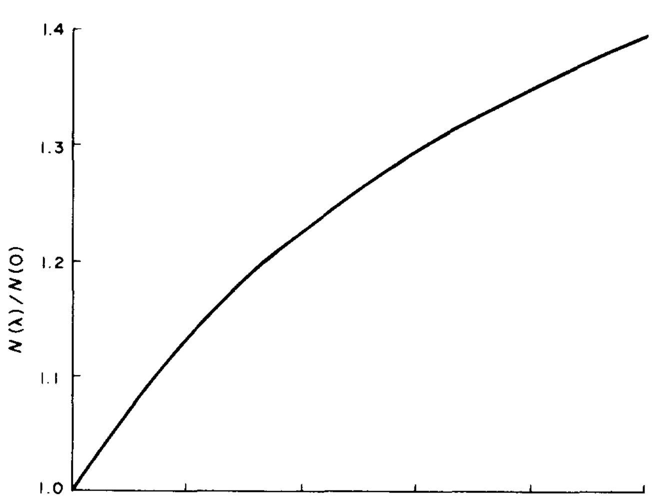

Figure 1. Graph of $N(\lambda)/N(0)$ vs λ for $0 \le \lambda \le 1$ and $h = 0.25$.

Table 3.

n	11	15	21	25	29
$N(0)/\mu\Omega$	6.233	6.264	6.272	6.273	6.273

Table 4.

n	11	15	21	25	29
$N(\infty)/\mu\Omega$	44.01	42.06	41.75	41.73	41.72

equation for the two-disk capacitor). In table 3 the first row gives the number of pivotal points and the second the computed value of $N(\lambda)/\mu\Omega$ for $\lambda = 0$ and $h = 0.05$. Convergence is seen to be achieved for $n = 25$. In table 4 similar results are given for $\lambda = \infty$, $h = 0.05$. The rate of convergence is seen to be much slower, but the torque, correct to 3 significant figures, can be confidently given as 41.7.

In the last case the torque on the disk can also be estimated from an asymptotic formula due to Leppington & Levine (1970). Adapting their notation to the problem under consideration, we have that

$$\frac{N(\infty)}{\mu\Omega} = \frac{\pi}{2h} - 2\log h + 2\left(\log 8\pi - \frac{5}{3}\right) + \frac{3}{\pi} h(\log h)^2 + \cdots, \text{ as } h \to 0. \qquad [42]$$

When $h = 0.05$, [42] gives

$$N(\infty)/\mu\Omega = 40.95,$$

the final term in [42] making a contribution of 0.43. Thus, the asymptotic result would seem to be reasonably close to the computed value.

6. SURFACE-VELOCITY DISTRIBUTION

As indicated in the introduction, the measurement of the azimuthal velocity of surface particles may prove to be a better experimental method for determining surface shear viscosities than torque measurements. Thus, we now discuss the computation of surface velocity for varying λ and h.

The surface velocity $v(\rho, -h)$ is given by

$$v(\rho, -h) = \pi \int_0^1 \sigma^*(\rho', 0) G_1^{(1)}(\rho, -h; \rho', 0)\rho' \, d\rho'; \qquad [43]$$

thus, from [17] and [20],

$$v(\rho, -h) = 2\Omega \int_0^1 f(\rho') \left\{ \int_0^\infty \frac{e^{-ht}}{1+\lambda t} J_1(t\rho) J_1(t\rho') \, dt \right\} d\rho', \qquad [44]$$

where $f(\rho')$ is related to $\theta(x)$ by [24]. Substituting from [24] in [44], the ρ'-integration can be carried out by first integrating by parts and then using the formula

$$\int_0^x \frac{\rho' J_0(t\rho')}{(x^2 - \rho'^2)^{1/2}} \, d\rho' = \frac{\sin tx}{t}. \qquad [45]$$

It is found that

$$v(\rho, -h) = \frac{4\Omega}{\pi} \int_0^1 \theta(x) \left\{ \int_0^\infty \frac{e^{-ht}}{1+\lambda t} J_1(t\rho) \sin tx \, dt \right\} dx. \qquad [46]$$

To compute the surface velocity profile the double integral in [46] must be evaluated from a knowledge of $\theta(x)$ at the pivotal points. The integrand of the infinite integral contains oscillatory terms, and can be converted into a more rapidly convergent form as follows. Write

$$\frac{1}{1+\lambda t}=\int_0^\infty \mathrm{e}^{-(\lambda t+1)\nu}\,\mathrm{d}\nu; \qquad [47]$$

then, interchanging orders of integration,

$$\int_0^\infty \frac{\mathrm{e}^{-ht}}{1+\lambda t}J_1(t\rho)\sin tx\,\mathrm{d}t=\int_0^\infty \mathrm{e}^{-\nu}\left\{\int_0^\infty \mathrm{e}^{-(\lambda\nu+h)t}J_1(t\rho)\sin tx\,\mathrm{d}t\right\}\mathrm{d}\nu. \qquad [48]$$

The inner integral in [48] can be evaluated using the result (Watson 1966) that

$$\int_0^\infty \mathrm{e}^{-at}J_1(bt)\,\mathrm{d}t=\frac{1}{b}-\frac{a}{b(a^2+b^2)^{1/2}},$$

where Re $a>0$, $\mathrm{Re}\,(a\pm ib)>0$, and the value of the square root taken is that which makes $|a+(a^2+b^2)^{1/2}|>|b|$. The final form obtained for $v(\rho,-h)$ is

$$v(\rho,-h)=\frac{2^{3/2}\Omega}{\pi}\int_0^1 \theta(x)P(x,\rho)\,\mathrm{d}x, \qquad [49]$$

where

$$P(x,\rho)=\int_0^\infty \mathrm{e}^{-\nu}\{x(r+u^2-x^2+\rho^2)^{1/2}-u(r-u^2+x^2-\rho^2)^{1/2}\}r^{-1}\,\mathrm{d}\nu, \qquad [50]$$

with $u=\lambda\nu+h$, and $r=\{(u^2-x^2+\rho^2)^2+4x^2u^2\}^{1/2}$. For given ρ and x the integral [50] converges rapidly and is computed by means of a NAG algorithm for the evaluation of integrals of the form

$$\int_0^\infty \mathrm{e}^{-\nu}g(\nu)\,\mathrm{d}\nu.$$

The finite integration in [49] is then completed using Simpson's rule.

In figure 2 we sketch some computed velocity profiles for $\lambda=0, 0.025, 0.075, 0.25, 0.5, 0.75$ and 1.00, with $h=0.25$. The velocity profiles flatten quite rapidly with increasing λ, as a result of the rigidity given to the surface by surfactants of increasingly large surface shear viscosity.

It was observed empirically that for $h=0.25$ and λ varying over a considerable range, the maximum value of $\Omega^{-1}v(\rho,-h)$ occurred at about $\rho=0.88$. This variation of the maximum is depicted in figure 3 for λ in the range [0, 1], and sample values are given in table 5, where values of ΩT are also given, T being the periodic time of revolution of a surface fluid particle with $\rho=0.88$. From table 5 it can be seen that there is about a 3.5 per cent decrease in $v_{\max}$ relative to a

Table 5.

λ	0	0.025	0.05	0.075	0.1	0.25	0.5	
$\Omega^{-1}v_{\max}$	0.692	0.668	0.647	0.629	0.612	0.532	0.444	
ΩT	7.990	8.277	8.546	8.790	9.035	10.393	12.453	
λ	0.75	1	1.25	1.5	1.75	2	3	5
$\Omega^{-1}v_{\max}$	0.384	0.339	0.304	0.276	0.252	0.233	0.177	0.118
ΩT	14.399	16.310	18.188	20.033	21.941	23.730	31.238	46.858

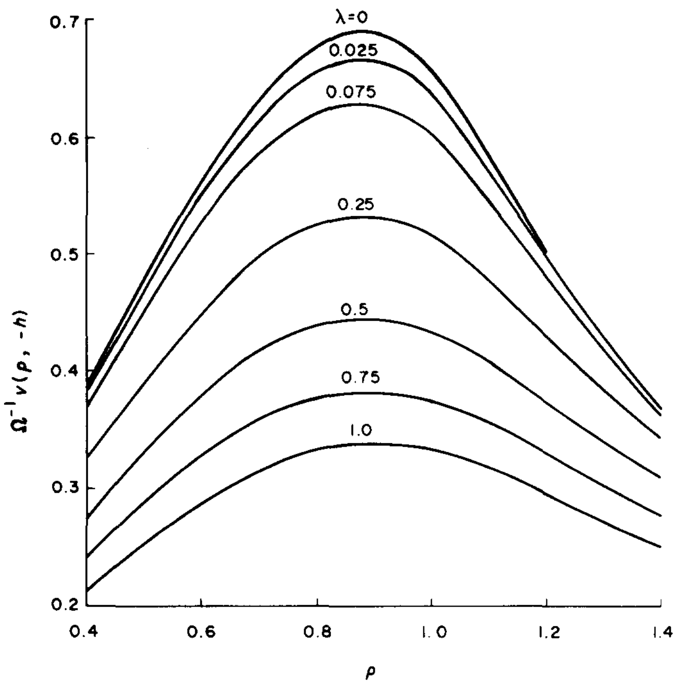

Figure 2. Velocity profiles for $\lambda = 0, 0.025, 0.075, 0.25, 0.5, 0.75$ and 1.00 with $h = 0.25$.

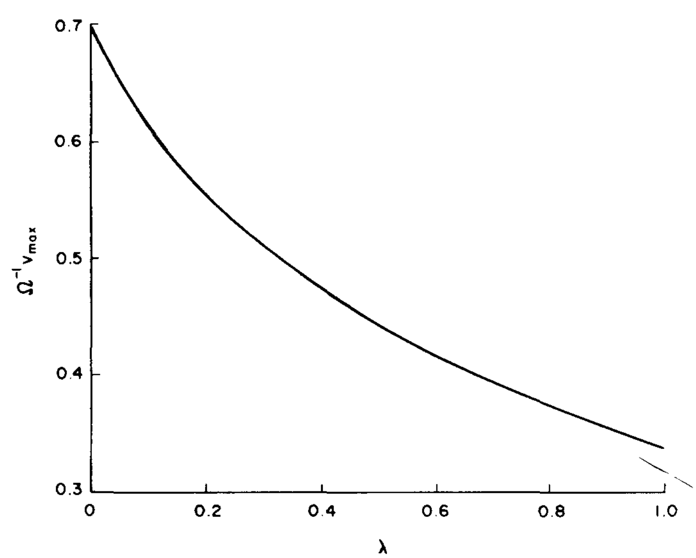

Figure 3. Graph of $\Omega^{-1}v_{\max}$ vs λ for $0 \leq \lambda \leq 1$ and $h = 0.25$ ($v_{\max}$ is computed at $\rho = 0.88$).

free surface for $\lambda = 0.025$, whereas for $\lambda = 0.1$ the percentage decrease is as much as 11.5 per cent.

Using the method of least squares the results in table 5 have been fitted to cubic polynomals for the ranges $0 \leq \lambda \leq 0.1$ and $0.1 \leq \lambda \leq 2$, corresponding approximately to the ranges $7.99 \leq \Omega T \leq 9.1$ and $9.1 \leq \Omega T \leq 23.73$. For an aqueous substrate with $\mu = 10^{-2}$ poise, the following formulae for η are obtained:

$$\eta = 10^{-2}\{-0.008114\Omega^3T^3 + 0.217908\Omega^2T^2 - 1.848144\Omega T + 4.99412\}, \quad 0 \leq \eta \leq 10^{-3}, \qquad [51]$$

$$\eta = 10^{-3}\{-0.0006\Omega^3T^3 + 0.03673\Omega^2T^2 + 0.60422\Omega T - 7.0420\}, \quad 10^{-3} \leq \eta \leq 2 \times 10^{-2}. \qquad [52]$$

The maximum error in [51] and [52] is about 1.5 per cent.

We end this section by estimating some typical periodic times of revolution of surface particles. The disk is taken to be of radius 1 cm, at a depth of 0.25 cm below the surface, and the substrate fluid is water. Then $\lambda = 100\,\eta$, where η is measured in surface poise. Consider a surfactant whose coefficient of surface shear viscosity is of the order of 10^{-3} surface poise, e.g. the stearic acid monolayers used by Poskanzer & Goodrich (1975). Then $\lambda \sim 0.1$ and from table 5, again focussing attention on a fluid particle at $\rho = 0.88$ cm, the period of rotation is T, where

$$T = \frac{2\pi \times 0.88}{0.612\Omega} = 9.03\Omega^{-1}\ \text{sec}.$$

For a clean surface ($\lambda = 0$) the corresponding period of rotation is $T_0 = 7.99\Omega^{-1}$ sec, there being about a 13 per cent increase in the period due to the presence of the surfactant. For $\Omega = 0.1$, giving an angular Reynolds number Re based on the disk radius of 10, the period of disk rotation T_d is about 1 min, and $T - T_0 \sim 10$ sec, whereas if $\Omega = 0.01$, Re = 1, $T_d \sim 10$ min and $T_1 - T_0 \sim 100$ sec. For a surfactant with $\eta \sim 10^{-4}$ surface poise, the corresponding time differences are reduced by a factor of approx. 10, the percentage increase in period due to the surfactant being about 1.5 per cent.

7. CONCLUDING REMARKS

In this paper a start has been made on the investigation of a number of axisymmetric viscometry problems in which an immersed body of revolution rotates slowly in a fluid whose surface is covered by an adsorbed film. The particular case of a thin disk fully immersed in a semi-infinite body of fluid has been examined in detail, but the formulation given in sections 2 and 3 is general enough to permit consideration of everywhere bounded fluids; all that is necessary is to use the Green's function appropriate to the container. The disk may also be replaced by a sphere, and some aspects of this problem are treated in a further paper in this series. For a sphere rotating in a semi-infinite fluid, the problem is best solved using a bispherical coordinate system, but it may also be formulated in terms of Fredholm integral equations of the first and second kinds. The latter approach appears to be the only one generally available for bounded fluids.

It is hoped that the analysis in this paper will stimulate experimental investigation of the techniques suggested for measuring the coefficient of surface shear viscosity. In particular it seems plausible that the submerged-disk technique will be superior to the Goodrich configuration for the measurement of small shear viscosities.

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